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The projector method for regular Poisson manifolds

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Abstract. Degenerate Lagrangians have recently been studied through a pair of projection operators induced by an almost-product structure [10]. We extend the projector method to regular Poisson manifolds. This approach proves to be useful in the interpretation of the theories of collective motion of N-particle systems for applications in nuclear physics.

1. Introduction

Some authors have made progress in the study of constrained systems in a modern spirit using projectors induced by almost-product structures ([3, 6, 9, 11, 12] and references therein). This technique goes back to Bhaskara and Wiswanath [3] in solving the problem of a global formulation of the local description of Dirac's theory of constraints (see Sudarshan and Mukunda [14], for instance) and also to de León and Rodrigues [8] in the global study of the dynamics of singular (autonomous and non-autonomous) Lagrangian functions on tangent bundles. The projector approach was applied recently to constraints defined by singular Lagrangian systems [10] (for an early local study of constraints in the framework of projection operators we refer to the papers [1, 2]).

One of the purposes of this article is to cover the subject in the more general context of a manifold endowed with a Poisson structure of constant rank. As we shall see, in the particular case of second-class constraints, the projectors are naturally defined by the constraint functions. The first-class case is more difficult since we need to fix the gauge but we only have the Poisson tensor and no other structure. A classical procedure [15] is to try to choose new constraints functions $\boldsymbol{f}^a$ such that the matrix $(\{y^a, \boldsymbol{f}^b\})$ is invertible, where $\{y^a\}$ are first class and so we may reproduce the projector method.

The other purpose is to give an example such that the projector method is adequate to study the behaviour of some physical systems around critical values. We apply the projector method to separate the translation motion of the centre of mass in the N-particle system. N is taken as a parameter of the physical system. Then it is shown that in the particular case of equal masses, the problem has a symplectic formulation in the limit of very large N (i.e. when $N \to \infty$) corresponding to the independent-particle model.

The work is structured as follows. In section 2 we give some definitions for clarity and support for the next sections. Projectors and second-class constraints are examined

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in section 3. We give global and local conditions for the construction of the Dirac's bracket from a projection operator. We conclude with the example. All objects considered throughout the paper as manifolds, mappings, forms, vector fields, etc are of C^∞ class. The manifolds are finite dimensional, Hausdorff, paracompact, etc. The summation convention on repeated indices is adopted.

2. Preliminaries

We first recall that an *almost-product* (or *involutive*) *structure* on a manifold W is a tensor field $\boldsymbol{C}$ of type $(1, 1)$ on W such that $\boldsymbol{C}^2 = \boldsymbol{I}$. The manifold W endowed with an almost-product structure $\boldsymbol{C}$ is said to be an *almost-product manifold*. If we set

$$\boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{I} + \boldsymbol{C}) \qquad \boldsymbol{B} = \tfrac{1}{2}(\boldsymbol{I} - \boldsymbol{C})$$

then

$$\boldsymbol{A}^2 = \boldsymbol{A} \qquad \boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{0} \qquad \boldsymbol{B}^2 = \boldsymbol{B} \qquad \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{I}. \tag{1}$$

Conversely, if $(\boldsymbol{A}, \boldsymbol{B})$ is a pair of tensor fields of type $(1, 1)$ on W satisfying (1), then $\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$ is an almost-product structure on W. Thus this structure is characterized by the complementary pair of *projectors* $(\boldsymbol{A}, \boldsymbol{B})$, with $\boldsymbol{A} : TW \to \mathrm{Im}\boldsymbol{A} \subset TW$, $\boldsymbol{B} : TW \to \mathrm{Im}\boldsymbol{B} \subset TW$ such that $TW = \mathrm{Im}\boldsymbol{A} \oplus \mathrm{Im}\boldsymbol{B}$.

Let $(W, \boldsymbol{\Pi})$ be a Poisson manifold, where $\boldsymbol{\Pi}$ is the *Poisson's tensor field* (a skew-symmetric tensor field of type $(2, 0)$ on W verifying the Jacoby identity

$$\boldsymbol{\Pi}(\alpha, \boldsymbol{\Pi}(\beta, \gamma)) + \boldsymbol{\Pi}(\beta, \boldsymbol{\Pi}(\gamma, \alpha)) + \boldsymbol{\Pi}(\gamma, \boldsymbol{\Pi}(\alpha, \beta)) = 0).$$

The Poisson bracket is defined by $\boldsymbol{\Pi}(\mathrm{d}f, \mathrm{d}g) = \{f, g\}$, for all $f, g \in C^\infty(W)$, where $C^\infty(W)$ is the space of C^∞ functions on W.

A Poisson structure $\boldsymbol{\Pi}$ on a manifold W induces a bundle morphism $\sharp : T^\star W \to TW$ such that

$$\alpha\sharp(\beta) = \boldsymbol{\Pi}(\alpha, \beta)$$

where α and β are one forms on W. In particular, if $f, g \in C^\infty(W)$ then $\mathrm{d}f\sharp(\mathrm{d}g) = \boldsymbol{\Pi}(\mathrm{d}f, \mathrm{d}g) = \{f, g\}$.

For a one-form α on W we shall denote by $X_\alpha = \sharp(\alpha)$ the corresponding Hamiltonian vector field. In what follows we shall consider only regular Poisson manifolds, referred to as Poisson manifolds, for brevity. This means that the *characteristic space* $\mathrm{Im}\,\sharp_x = \sharp_x(T_x^\star W) \subset T_x W$ at x has the same dimension for all $x \in W$.

3. Projectors

Let $(W, \boldsymbol{\Pi})$ be a Poisson manifold, $\boldsymbol{Q} : TW \to TW$ a $(1, 1)$-tensor field on W and $\boldsymbol{Q}^\star$ the adjoint operator of $\boldsymbol{Q}$, i.e. $\boldsymbol{Q}^\star(\gamma) = \gamma \circ \boldsymbol{Q}$, for all one forms γ on W.

Proposition 3.1. If $\boldsymbol{Q} \circ \sharp = \sharp \circ \boldsymbol{Q}^\star$ then $\boldsymbol{\Pi}(\boldsymbol{Q}^\star(\alpha), \beta) = \boldsymbol{\Pi}(\alpha, \boldsymbol{Q}^\star(\beta))$.

Proof. Indeed,

$$\begin{aligned}\boldsymbol{\Pi}(\boldsymbol{Q}^\star(\alpha), \beta) &= \boldsymbol{Q}^\star(\alpha)\sharp(\beta) = \alpha(\boldsymbol{Q} \circ \sharp)\beta \\ &= \alpha\sharp(\boldsymbol{Q}^\star\beta) = \boldsymbol{\Pi}(\alpha, \boldsymbol{Q}^\star(\beta)).\end{aligned}$$

□

Now, $\sharp$ maps $T^\star W$ to TW and so the adjoint $\sharp^\star$ is also a map from $T^\star W$ to TW and in fact $\sharp^\star = -\sharp$ since $\mathbf{\Pi}$ is skew-symmetric. Therefore the assumption of proposition 3.1 says that $\boldsymbol{Q} \circ \sharp = -(\boldsymbol{Q} \circ \sharp)^\star$. We note that $\boldsymbol{Q}(X_\beta) = X_{\boldsymbol{Q}^\star \beta}$ and $\boldsymbol{Q}^\star(\alpha)(X_\beta) = -\boldsymbol{Q}^\star(\beta)(X_\alpha)$, since

$$\begin{aligned} &\boldsymbol{Q}^\star(\alpha)(X_\beta) = \boldsymbol{Q}^\star(\alpha)\sharp(\beta) = \mathbf{\Pi}(\boldsymbol{Q}^\star(\alpha), \beta) \\ &-\boldsymbol{Q}^\star(\beta)(X_\alpha) = -\boldsymbol{Q}^\star(\beta)\sharp(\alpha) = -\mathbf{\Pi}(\boldsymbol{Q}^\star(\beta), \alpha) = \mathbf{\Pi}(\boldsymbol{Q}^\star(\alpha), \beta). \end{aligned}$$

Proposition 3.2. Suppose that $\boldsymbol{Q}^\star$ is a projector, i.e. $\boldsymbol{Q}^\star \circ \boldsymbol{Q}^\star = (\boldsymbol{Q}^\star)^2 = \boldsymbol{Q}^\star$. If $\boldsymbol{Q} \circ \sharp = \sharp \circ \boldsymbol{Q}^\star$ then

$$\mathbf{\Pi}(\alpha, \boldsymbol{Q}^\star(\beta)) = \mathbf{\Pi}(\boldsymbol{Q}^\star(\alpha), \boldsymbol{Q}^\star(\beta)). \tag{2}$$

Furthermore, if $\boldsymbol{P}^\star = \boldsymbol{I} - \boldsymbol{Q}^\star$, then

$$\mathbf{\Pi}(\boldsymbol{P}^\star(\alpha), \boldsymbol{P}^\star(\beta)) = \mathbf{\Pi}(\alpha, \beta) - \mathbf{\Pi}(\boldsymbol{Q}^\star(\alpha), \beta) \qquad \forall \alpha, \beta. \tag{3}$$

In particular, if $\alpha = \mathrm{d}f$, $\beta = \mathrm{d}g$, where $f, g \in \mathcal{C}^\infty(W)$ then

$$\mathbf{\Pi}(\boldsymbol{P}^\star(\mathrm{d}f), \boldsymbol{P}^\star(\mathrm{d}g)) = \{f, g\} - \mathbf{\Pi}(\boldsymbol{Q}^\star(\mathrm{d}f), \mathrm{d}g). \tag{4}$$

Proof. We have

$$\mathbf{\Pi}(\alpha, \boldsymbol{Q}^\star(\beta)) = \mathbf{\Pi}(\alpha, (\boldsymbol{Q}^\star)^2(\beta)) = \mathbf{\Pi}(\alpha, \boldsymbol{Q}^\star[\boldsymbol{Q}^\star(\beta)]) = \mathbf{\Pi}(\boldsymbol{Q}^\star(\alpha), \boldsymbol{Q}^\star(\beta)).$$

The use of the bilinearity of $\mathbf{\Pi}$ and a very simple calculation shows (3) and (4). □

We remark that obsviously $\mathbf{\Pi}(\alpha, \boldsymbol{P}^\star(\beta)) = \mathbf{\Pi}(\boldsymbol{P}^\star(\alpha), \boldsymbol{P}^\star(\beta))$ and also that the tensor field $\mathbf{\Pi}(\boldsymbol{P}^\star(\alpha), \boldsymbol{P}^\star(\beta))$ is a Poisson tensor iff the distribution $\mathrm{Im}\, \boldsymbol{P}$ is involutive (see [3]). This is the case when there is defined a pre-symplectic form on W such that $\mathrm{Ker}\, \omega = \mathrm{Im}\, \boldsymbol{P}$ (see [10]).

Suppose that K is an embedded manifold of W, locally characterized by a coordinate system (y^i), $i \in \{1, \ldots, \dim W\}$, defined on a neighbourhood $V \subset W$ of some point z of $K \subset W$ such that $y = (y^a)|_{U=K\cap V} \equiv 0$, $a \in \{1, \ldots, \dim W - \dim K\}$, are *independent second-class constraints* (the set $\{\mathrm{d}y^a\}$ is linearly independent) and the matrix with elements $\lambda^{ab} = \{y^a, y^b\}$ is non-singular and skew-symmetric (thus the number of contraints are even). Then the set $\{X^a = \sharp(\mathrm{d}y^a)\}$ of Hamiltonian vector fields is linearly independent. As usual, $\mathrm{d}y^a(X^b) = \mathrm{d}y^a(\sharp(\mathrm{d}y^b)) = \{y^a, y^b\} = \mathbf{\Pi}(\mathrm{d}y^a, \mathrm{d}y^b)$.

The functions y^a and the Hamiltonian vector fields X^a suggest the definition of the following tensor field

$$\boldsymbol{Q} = \lambda_{ab}\, \mathrm{d}y^b \otimes X^a \tag{5}$$

where λ_{ab} are a set of smooth functions on W.

We now examine the conditions for such a $\boldsymbol{Q}$ to be locally a *projector*, verifying the *hypothesis of proposition 3.1*. Let us set $M_{ad} = \lambda_{ab}\lambda^{bc}\lambda_{cd}$. As

$$\boldsymbol{Q}^\star(\alpha) = [\alpha(X^a)\lambda_{ab}]\, \mathrm{d}y^b \qquad \boldsymbol{Q}^\star(\mathrm{d}y^b) = [\lambda^{ba}\lambda_{ac}]\, \mathrm{d}y^c \tag{6}$$

we have

$$\begin{aligned} (\boldsymbol{Q}^\star)^2(\alpha) &= \boldsymbol{Q}^\star(\alpha(X^a)\lambda_{ab}\, \mathrm{d}y^b) = [\alpha(X^a)\lambda_{ab}]\boldsymbol{Q}^\star(\mathrm{d}y^b) \\ &= \alpha(X^a)[\lambda_{ab}\lambda^{bc}\lambda_{cd}]\, \mathrm{d}y^d = [\alpha(X^a)M_{ad}]\, \mathrm{d}y^d. \end{aligned}$$

If we suppose now that the functions λ_{ab} are elements of the inverse matrix of (λ^{ab}), that is, $\lambda^{ba}\lambda_{ac} = \delta^b_c$ then $M_{ab} = \lambda_{ab}$ and so $(\boldsymbol{Q}^\star)^2(\alpha) = (\boldsymbol{Q}^\star)(\alpha)$ for all α.

Now,

$$(\boldsymbol{Q} \circ \sharp)(\alpha) = \boldsymbol{Q}(X_\alpha) = [\mathrm{d}y^b(X_\alpha)\lambda_{ab}]X^a,$$
$$(\sharp \circ \boldsymbol{Q}^\star)(\alpha) = \sharp(\alpha(X^b)\lambda_{ba}\,\mathrm{d}y^a) = \alpha(X^b)\lambda_{ba}\sharp(\mathrm{d}y^a) = [\alpha(X^b)\lambda_{ba}]X^a$$

and

$$\mathrm{d}y^b(X_\alpha) = \mathrm{d}y^b\sharp(\alpha) = -\alpha\sharp(\mathrm{d}y^b) = -\alpha(X^b) \tag{7}$$

and since (λ_{ba}) is skew-symmetric it follows that $\boldsymbol{Q} \circ \sharp = \sharp \circ \boldsymbol{Q}^*$.

Consider the particular case where the forms are exact, say $\alpha = \mathrm{d}f$, $\beta = \mathrm{d}g$ and set

$$\boldsymbol{\Pi}(\boldsymbol{P}^\star\mathrm{d}f, \boldsymbol{P}^\star\mathrm{d}g) \overbrace{=}^{\mathrm{def}} \{f, g\}_{\boldsymbol{D}}.$$

Then it is straightforward to show that

$$\{f, g\}_{\boldsymbol{D}} = \{f, g\} - \{f, y^a\}\lambda_{ab}\{y^b, g\}$$

which is the *modified Poisson's bracket* introduced by Dirac to deal with second-class constraints in his theory.

We close this section with the following remarks: (a) The tensor field $\boldsymbol{\Pi}(\boldsymbol{Q}^\star, \boldsymbol{Q}^\star)$ is locally a bivector of the following form

$$\tfrac{1}{2}\lambda_{ab}X^a \wedge X^b \tag{8}$$

since

$$\begin{aligned}\boldsymbol{\Pi}(\boldsymbol{Q}^\star\alpha, \boldsymbol{Q}^\star\beta) &= \alpha(X^b)\lambda_{ba}\,\mathrm{d}y^a(X_\beta)\\ &= \lambda_{ab}\alpha(X^b)\beta(X^a) = \tfrac{1}{2}\lambda_{ab}X^a \wedge X^b(\alpha, \beta).\end{aligned}$$

(b) If K is locally characterized by first-class functions $\{y^a\}$, that is $\{y^a, y^b\}$ vanishes on K, then a classical procedure is to choose new constraints functions $\boldsymbol{f}^a$ such that the matrix with entries $\theta^{ab} = \{y^a, \boldsymbol{f}^b\}$ is invertible. If we set

$$\boldsymbol{Q} = \theta_{ab}\,\mathrm{d}\boldsymbol{f}^b \otimes X^a$$

then we may reproduce the above procedure. The mixed (first- and second-class) case is obtained in a similar way [10].

4. An example: The centre of mass of the N-particles system

Most equations describing the behaviour of physical systems contain parameters. Of particular interest is the behaviour of solutions when such a parameter approaches a critical value, usually zero or infinity [16]. In this example we shall see that the projector method proves to be useful in the interpretation of the theories of collective motion of N-particle systems for applications in nuclear physics (we shall consider N as the parameter, the number of intrinsical particles).

In such theories it is assumed that the system can be separated in collective mode, described by collective coordinates $(\boldsymbol{q}, \boldsymbol{p})$ and independent particle motions described by intrinsic coordinates $(q_1, \ldots, q_{3(N-1)}, p_1, \ldots, p_{3(N-1)})$. This is obtained by introducing suitable constraints on the original $6N$ coordinates of the phase space. For more details see [7, 13] and references therein.

So, let us consider a system of N-particles each with unit mass. The phase space relative to the centre of mass is the standard symplectic manifold $\boldsymbol{M} = (\mathbb{R}^{6N}, \omega)$, whose coordinates are subjected to the second-class constraints

$$\phi_a^1 = \sum_{i=1}^{N} \boldsymbol{r}_{ia} = 0 \qquad a = x, y, z \qquad \phi_a^2 = \sum_{i=1}^{N} \boldsymbol{p}_{ia} = 0 \qquad a = x, y, z.$$

Here $\boldsymbol{r}_{ia}$ and $\boldsymbol{p}_{ia}$ are the cartesian components of the vectors $\boldsymbol{r}$ and $\boldsymbol{p}$ in $\mathbb{R}^3$. In the absence of external forces the Hamiltonian is

$$\mathcal{H} = \sum_i \frac{\boldsymbol{p}_i^2}{2} - \sum_{i<j} V(|\boldsymbol{r}_i - \boldsymbol{r}_j|).$$

The associated Hamiltonian vector field is given by

$$X_{\mathcal{H}} = \sharp(d\mathcal{H}) = \sum_{i,a} \left[\boldsymbol{p}_{ia} \frac{\partial}{\partial \boldsymbol{r}_{ia}} + \frac{\partial V(|\boldsymbol{r}_i - \boldsymbol{r}_j|)}{\partial \boldsymbol{r}_{ia}} \frac{\partial}{\partial \boldsymbol{p}_{ia}} \right]$$

and it is easy to see that

$$\boldsymbol{\Pi} = \sum_a \sum_i^N \frac{\partial}{\partial \boldsymbol{p}_{ia}} \wedge \frac{\partial}{\partial \boldsymbol{r}_{ia}}$$

is the standard Poisson tensor, whose representation is given by the $6N \times 6N$ matrix

$$\boldsymbol{\Pi} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0} \end{pmatrix}$$

where $\boldsymbol{0}$ and $\boldsymbol{I}$ are $3N \times 3N$ null and unit matrices respectively. In order to deal with this constrained system by preserving the symmetry between the particles we use the projector method. We have,

$$\mathrm{d}\phi_a^1 = \sum_i^N \mathrm{d}\boldsymbol{r}_{ia} \qquad \mathrm{d}\phi_a^2 = \sum_i^N \mathrm{d}\boldsymbol{p}_{ia}$$

$$X_a^1 = \sharp(\mathrm{d}\phi_a^1) = -\sum_i \frac{\partial}{\partial \boldsymbol{p}_{ia}} \qquad X_a^2 = \sharp(\mathrm{d}\phi_a^2) = \sum_i \frac{\partial}{\partial \boldsymbol{r}_{ia}}.$$

According to (5) the projector $\boldsymbol{Q}$ is written as

$$\boldsymbol{Q} = \lambda_{AB}\, \mathrm{d}\phi^B \otimes X^A \qquad A, B = 1, 2.$$

As the constraints are second class, the functions λ_{AB} are the matrix elements of the inverse of the matrix λ defined by

$$\lambda_a^{11} = \lambda_a^{22} = (\mathrm{d}\phi_a^A[X_a^A]) = 0$$
$$\lambda_a^{12} = -\lambda_a^{21} = (\mathrm{d}\phi_a^1[X_b^2]) = -(\mathrm{d}\phi_a^2[X_b^1]) = -\delta_{ab} \sum_{ij} \delta_{ij} = -N$$

that is

$$\lambda^{-1} = \frac{1}{N} \begin{pmatrix} \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-1} & \boldsymbol{0} \end{pmatrix}.$$

Therefore the projector is given by

$$\boldsymbol{Q} = \frac{1}{N}\sum_{ij,a}\left(\mathrm{d}\boldsymbol{r}_{ja}\otimes\frac{\partial}{\partial\boldsymbol{r}_{ia}} + \mathrm{d}\boldsymbol{p}_{ja}\otimes\frac{\partial}{\partial\boldsymbol{p}_{ia}}\right)$$

or

$$\boldsymbol{Q} = \begin{pmatrix} \mathcal{Q} & \mathbf{0} \\ \mathbf{0} & \mathcal{Q} \end{pmatrix}$$

where $\mathcal{Q}$ is the $3N \times 3N$ matrix

$$\mathcal{Q} = \frac{1}{N}\begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & \cdots & 1 \end{pmatrix}.$$

It is easy to see that $\mathcal{Q}^2 = \mathcal{Q}$ and then $\boldsymbol{Q}^2 = \boldsymbol{Q}$. The complementary projetor is

$$\boldsymbol{P} = \sum_{ij,a}\left(\delta_i^j - \frac{1}{N}\right)\left(\mathrm{d}\boldsymbol{r}_{ja}\otimes\frac{\partial}{\partial\boldsymbol{r}_{ia}} + \mathrm{d}\boldsymbol{p}_{ja}\otimes\frac{\partial}{\partial\boldsymbol{p}_{ia}}\right)$$

or in matrix representation

$$\boldsymbol{P} = \boldsymbol{I}_{6N} - \boldsymbol{Q} = \begin{pmatrix} \mathcal{P} & \mathbf{0} \\ \mathbf{0} & \mathcal{P} \end{pmatrix}$$

where $\mathcal{P}$ is the $3N \times 3N$ matrix

$$\mathcal{P} = \frac{1}{N}\begin{pmatrix} N-1 & -1 & . & . & . & -1 \\ -1 & N-1 & -1 & . & . & . \\ . & -1 & N-1 & -1 & . & . \\ . & . & -1 & N-1 & -1 & . \\ . & . & . & -1 & N-1 & -1 \\ -1 & . & . & . & -1 & N-1 \end{pmatrix}.$$

The constrained tensor $\boldsymbol{\Pi}_{\boldsymbol{P}} = \boldsymbol{\Pi}(\boldsymbol{P}^\star, \boldsymbol{P}^\star)$, is

$$\boldsymbol{\Pi}_{\boldsymbol{P}} = \sum_{ij,a}\left(\delta_i^j - \frac{1}{N}\right)\left(\frac{\partial}{\partial\boldsymbol{p}_{ia}}\wedge\frac{\partial}{\partial\boldsymbol{r}_{ja}}\right)$$

or in matrix representation

$$\boldsymbol{\Pi}_{\boldsymbol{P}} = \begin{pmatrix} \mathbf{0} & \mathcal{P} \\ -\mathcal{P} & \mathbf{0} \end{pmatrix}.$$

This Poisson tensor gives rise to the following Poisson brackets restricted to the $6(N-1)$ dimensional symplectic leaf defined by the constraints:

$$\begin{aligned} &\{\boldsymbol{r}_{ia}, \boldsymbol{r}_{jb}\} = \{\boldsymbol{p}_{ia}, \boldsymbol{p}_{jb}\} = 0 \\ &\{\boldsymbol{r}_{jb}, \boldsymbol{p}_{ia}\} = \delta_b^a\left(\delta_j^i - \frac{1}{N}\right) \qquad a, b = x, y, z. \end{aligned} \tag{9}$$

The dynamics on the leaf is determined by the tangent Hamiltonian vector field $\overline{X}_{\mathcal{H}}$ obtained by the projection of $X_{\mathcal{H}}$

$$\sharp\boldsymbol{P}^\star(d\mathcal{H}) = \overline{X}_{\mathcal{H}} = \sum_{i,a}\left[(\boldsymbol{p}_{ia} - \langle\boldsymbol{p}_a\rangle)\frac{\partial}{\partial\boldsymbol{r}_{ia}} + \left(\frac{\partial V(|\boldsymbol{r}_i - \boldsymbol{r}_j|)}{\partial\boldsymbol{r}_{ia}} - \langle F_a\rangle\right)\frac{\partial}{\partial\boldsymbol{p}_{ia}}\right]$$

where

$$\langle \boldsymbol{p}_a \rangle = \frac{1}{N} \sum_i \boldsymbol{p}_{ia}, \langle F_a \rangle = \frac{1}{N} \sum_i \frac{\partial V(|\boldsymbol{r}_i - \boldsymbol{r}_j|)}{\partial \boldsymbol{r}_{ia}}.$$

From equation (9) we see that when $N \to \infty$ the system is free of constraints (indeed the problem has a symplectic formulation). This means that only for a large number of particles the independent-particle model is a good approximation [13]. This phenomenon is common in mechanical systems described by Poisson manifolds as was pointed out in [16]: when a physical parameter in the system reaches a limiting value (usually 0 or ∞) the limiting system has a symplectic formulation.

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